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THOMSON REUTERS

[Mallika, 6(7): July 2019]
DOI- 10.5281/zenodo.3351447

ISSN 2348 – 8034
Impact Factor- 5.070

GLOBAL JOURNAL OF ENGINEERING SCIENCE AND RESEARCHES

ON THE NEGATIVE PELL EQUATION

$$y^2 = 23x^2 - 11$$

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ABSTRACT

The binary quadratic Diophantine equation represented by the negative pellian $y^2 = 23x^2 - 11$ is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.

Keywords: *Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.*

I. INTRODUCTION

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-10]. In this communication, yet another interesting equation given by $y^2 = 23x^2 - 11$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The negative Pell equation representing hyperbola under consideration is,

$$y^2 = 23x^2 - 11 \tag{1}$$

The smallest negative integer solutions of (1) are,

$x_0 = 2$, $y_0 = 9$

The pellian equation is

$$y^2 = 23x^2 + 1 \tag{2}$$

The initial solution of pellian equation is

$\tilde{x}_0 = 5, \tilde{y}_0 = 24$

The general solution (x_n, y_n) of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{23}} g_n \, , \quad \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = \left(24 + 5\sqrt{23}\right)^{n+1} - \left(24 - 5\sqrt{23}\right)^{n+1}$$

$$g_n = \left(24 + 5\sqrt{23}\right)^{n+1} + \left(24 - 5\sqrt{23}\right)^{n+1} \quad n = 0,1,2,.....$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$x_{n+1} = f_n + \frac{9}{2\sqrt{23}} g_n$$

$$y_{n+1} = \frac{9}{2} f_n + \frac{23}{\sqrt{23}} g_n$$

The recurrence relation satisfied by the solution x and y are given by,

$$x_{n+3} - 48x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 48y_{n+2} + y_{n+1} = 0 \qquad n=0,1,2,3\ldots..$$

Some numerical examples of x and y satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	x_n	y_n
0	2	9
1	93	446
2	4462	21399
3	214083	1026706

From the above table, we observe some interesting relations among the solutions which are presented below.

x_n values are even and odd alternatively.

y_n values are odd and even alternatively.

Each of the following expression is a nasty number:

$$\frac{6}{55}\left[110 + 892x_{2n+2} - 18x_{2n+4}\right]$$

$$\frac{6}{440}\left[880 + 7133x_{2n+2} - 3x_{2n+4}\right]$$

$$\frac{6}{44}\left[88 + 713x_{2n+2} - 3y_{2n+3}\right]$$

$$\frac{6}{12661}\left[25322 + 205252x_{2n+2} - 18y_{2n+4}\right]$$

$$\frac{1}{11}\left[132 + 23x_{2n+4} - 223y_{2n+3}\right]$$

$$\frac{6}{12661}\left[25322 + 92x_{2n+4} - 42798y_{2n+3}\right]$$

$$\frac{6}{1265}\left[2530 + 92y_{2n+3} - 4278y_{2n+2}\right]$$

[Mallika, 6(7): July 2019] ISSN 2348 – 8034
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$$\frac{1}{220}\left[2640+2y_{2n+4}-4462y_{2n+3}\right]$$

$$\frac{6}{55}\left[110+42798x_{2n+3}-892x_{2n+4}\right]$$

$$\frac{6}{11}\left[22+4278x_{2n+3}-892y_{2n+3}\right]$$

$$\frac{1}{11}\left[132+223y_{2n+4}-51313x_{2n+3}\right]$$

$$\frac{6}{44}\left[88+713x_{2n+4}-7133y_{2n+3}\right]$$

$$\frac{6}{11}\left[22+205252x_{2n+4}-42798y_{2n+4}\right]$$

$$\frac{6}{1265}\left[2530+4278y_{2n+4}-205252y_{2n+3}\right]$$

Each of the following expressions is a cubical integer.

$$\frac{1}{55}\left[892x_{3n+3}-18x_{3n+5}+2676y_{n+1}-54x_{n+1}\right]$$

$$\frac{1}{440}\left[7133x_{3n+3}-3x_{3n+5}+21399x_{n+1}-9x_{n+3}\right]$$

$$\frac{1}{12661}\left[205252x_{3n+3}-18y_{3n+5}+615756x_{n+1}-54y_{n+3}\right]$$

$$\frac{1}{44}\left[713x_{3n+3}-3y_{3n+4}+2139x_{n+1}-9y_{n+2}\right]$$

$$\frac{1}{66}\left[23x_{3n+5}-223y_{3n+4}+69x_{n+2}-669y_{n+1}\right]$$

$$\frac{1}{12661}\left[92x_{3n+5}-42798y_{3n+4}+276x_{n+3}-128394y_{n+1}\right]$$

$$\frac{1}{1265}\left[92y_{3n+4}-4278y_{3n+3}+276y_{n+2}-12834y_{n+1}\right]$$

$$\frac{1}{660}\left[y_{3n+5}-2231y_{3n+4}+3y_{n+3}-6693y_{n+1}\right]$$

$$\frac{1}{55}\left[42798x_{3n+4}-892x_{3n+5}+128394x_{n+2}-2676x_{n+3}\right]$$

$$\frac{1}{11}\left[4278x_{3n+4}-892y_{3n+4}+12834x_{n+2}-2676y_{n+2}\right]$$

$$\frac{1}{66}\left[223y_{3n+5}-51313x_{3n+4}+669y_{n+3}-153939x_{n+2}\right]$$

$$\frac{1}{44}\left[713x_{3n+5}-7133y_{3n+4}+2139x_{n+3}-21399y_{n+2}\right]$$

$$\frac{1}{11}\left[205252x_{3n+5}-42798y_{3n+5}+615756x_{n+3}-128394y_{n+3}\right]$$

$$\frac{1}{1265}\left[4278y_{3n+5}-205252y_{3n+4}+12834y_{n+3}-615756y_{n+2}\right]$$

Each of the following expressions is a biquadratic integer.

$$\frac{1}{55}\left[892x_{4n+4}-18x_{4n+6}+3568x_{2n+2}-72x_{2n+4}+330\right]$$

$$\frac{1}{440}\left[7133x_{4n+4}-3x_{4n+6}+28532x_{2n+2}-12x_{2n+4}+2640\right]$$

$$\frac{1}{44}\left[713x_{4n+4}-3y_{4n+5}+2852x_{2n+2}-12y_{2n+3}+264\right]$$

$$\frac{1}{12661}\left[\begin{array}{l}205252x_{4n+4}-18y_{4n+6}+821008x_{2n+2}-72y_{2n+4}\\+75966\end{array}\right]$$

$$\frac{1}{66}\left[23x_{4n+6}-223y_{4n+5}+92x_{2n+4}-892y_{2n+3}+396\right]$$

$$\frac{1}{12661}\left[92x_{4n+6}-42798y_{4n+5}+368x_{2n+4}-171192y_{2n+3}+75966\right]$$

$$\frac{1}{1265}\left[92y_{4n+5}-4278y_{4n+4}+36y_{2n+3}-17112y_{2n+2}+7590\right]$$

$$\frac{1}{660}\left[y_{4n+6}-2231y_{4n+5}+4y_{2n+4}-8924y_{2n+3}+2640\right]$$

$$\frac{1}{55}\left[42798x_{4n+5}-892x_{4n+6}+171198x_{2n+3}-3568x_{2n+4}+330\right]$$

$$\frac{1}{11}\left[12426y_{4n+5}-1242x_{4n+6}+49704y_{2n+3}-4968x_{2n+4}+936\right]$$

$$\frac{1}{66}\left[223y_{4n+6}-51313x_{4n+5}+892y_{2n+4}-205252x_{2n+3}+396\right]$$

$$\frac{1}{44}\left[713x_{4n+6}-7133y_{4n+5}+2852x_{2n+4}-28532y_{2n+3}+264\right]$$

$$\frac{1}{1265}\left[4278y_{4n+6}-205252y_{4n+5}+17112y_{2n+4}-821008y\ \right]$$

$$\frac{1}{11}\left[\begin{array}{l}205252x_{4n+6}-42798y_{4n+6}+821008x_{2n+4}-171192y_{2n+4}\\+66\end{array}\right]$$

Each of the following expression is a quintic integer:

$$\frac{1}{55}\begin{bmatrix}892x_{5n+5}-18x_{5n+7}+4460x_{3n+3}-90x_{3n+5}\\+8920x_{n+1}-180y_{n+1}\end{bmatrix}$$

$$\frac{1}{440}\begin{bmatrix}7133x_{5n+5}-3x_{5n+7}-356655x_{3n+3}-15x_{3n+5}\\+71330x_{n+1}-30x_{n+3}\end{bmatrix}$$

$$\frac{1}{44}\begin{bmatrix}713x_{5n+5}-3y_{5n+6}+3565x_{3n+3}-15y_{3n+4}\\+7130x_{n+1}-30y_{n+2}\end{bmatrix}$$

$$\frac{1}{12661}\begin{bmatrix}205252x_{5n+5}-18y_{5n+7}+1026260x_{3n+3}-90y_{3n+5}+\\2052520x_{n+1}-180y_{n+3}\end{bmatrix}$$

$$\frac{1}{66}\begin{bmatrix}23x_{5n+7}-223y_{5n+6}+115x_{3n+5}-1115y_{3n+4}\\+230x_{n+2}-2230y_{n+1}\end{bmatrix}$$

$$\frac{1}{12661}\begin{bmatrix}92x_{5n+7}-42798y_{5n+6}+460x_{3n+5}-213990y_{3n+4}\\+920x_{n+3}-427980y_{n+1}\end{bmatrix}$$

$$\frac{1}{1265}\begin{bmatrix}92y_{5n+6}-4278y_{5n+5}+460y_{3n+4}-21390y_{3n+3}\\+920y_{n+2}-42780y_{n+1}\end{bmatrix}$$

$$\frac{1}{660}\begin{bmatrix}y_{5n+7}-2231y_{5n+6}+5y_{3n+5}-11155y_{3n+4}\\+20y_{n+3}-22310y_{n+1}\end{bmatrix}$$

$$\frac{1}{55}\begin{bmatrix}42798x_{5n+6}-892x_{5n+6}+213990x_{3n+4}-4460x_{3n+5}\\+427980x_{n+2}-8920x_{n+3}\end{bmatrix}$$

$$\frac{1}{11}\begin{bmatrix}42798x_{5n+6}-892x_{5n+7}+21390x_{3n+4}-4460y_{3n+4}\\+42780x_{n+2}-8920y_{n+2}\end{bmatrix}$$

$$\frac{1}{66}\begin{bmatrix}223y_{5n+7}-51313x_{5n+6}+1115y_{3n+5}-256565x_{3n+4}\\+2230y_{n+3}-513130x_{n+2}\end{bmatrix}$$

$$\frac{1}{44}\begin{bmatrix}713x_{5n+7}-7133y_{5n+6}+3565x_{3n+5}-35665y_{3n+4}\\+7130x_{n+3}-71330y_{n+2}\end{bmatrix}$$

$$\frac{1}{11}\begin{bmatrix}205252x_{5n+7}-42798y_{5n+7}+1026260x_{3n+5}-213990y_{3n+5}\\+2052520x_{n+3}-427980y_{n+3}\end{bmatrix}$$

$$\frac{1}{1265}\begin{bmatrix}4278y_{5n+7}-205252y_{5n+6}+21390y_{3n+5}-1026260y_{3n+4}\\+42780y_{n+3}-2052520y_{n+2}\end{bmatrix}$$

Relations among the solutions are given below:

$x_{n+2} = 5y_{n+1} + 24x_{n+1}$

$x_{n+3} = 240y_{n+1} + 1151x_{n+1}$

$y_{n+2} = 24y_{n+1} + 115x_{n+1}$

$y_{n+3} = 1151y_{n+1} + 5520x_{n+1}$

$x_{n+3} = 48x_{n+2} - x_{n+1}$

$5y_{n+2} = 24x_{n+2} - x_{n+1}$

$5y_{n+3} = 1151x_{n+2} - 24x_{n+1}$

$10y_{n+2} = x_{n+3} - x_{n+1}$

$240y_{n+3} = 1151x_{n+3} - x_{n+1}$

$24y_{n+3} = 1151y_{n+2} + 115x_{n+1}$

$24x_{n+3} = 5y_{n+1} + 1151x_{n+2}$

$24y_{n+2} = y_{n+1} + 115x_{n+2}$

$24y_{n+3} = 24y_{n+1} + 5520x_{n+2}$

$1151y_{n+2} = 24y_{n+1} + 115x_{n+3}$

$1151y_{n+3} = y_{n+1} + 5520x_{n+3}$

$115y_{n+3} = 5520y_{n+2} - 115y_{n+1}$

$5y_{n+2} = x_{n+3} - 24x_{n+2}$

$5y_{n+3} = 24x_{n+3} - x_{n+2}$

$y_{n+3} = 24y_{n+2} + 115x_{n+2}$

$24y_{n+3} = y_{n+2} + 115x_{n+3}$

Remarkable Observation

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$Y^2 - 23X^2 = 484$	$\left(4y_{n+1} - 18x_{n+1}, 92x_{n+1} - 18y_{n+1}\right)$
2	$Y^2 - 23X^2 = 12100$	$\left(4x_{n+2} - 186x_{n+1}, 892x_{n+1} - 18x_{n+2}\right)$
3	$Y^2 - 23X^2 = 6969600$	$\left(2x_{n+3} - 4462x_{n+1}, 21399x_{n+1} - 9x_{n+3}\right)$

ISSN 2348 – 8034
Impact Factor- 5.070

4	$Y^2 - 23X^2 = 69696$	$\left(2y_{n+2} - 446x_{n+1},\ 2139x_{n+1} - 9y_{n+2}\right)$
5	$Y^2 - 23X^2 = 641203684$	$\left(4y_{n+3} - 42798x_{n+1},\ 205252x_{n+1} - 18y_{n+3}\right)$
6	$Y^2 - 23X^2 = 69696$	$\left(93y_{n+1} - 9x_{n+2},\ 46x_{n+2} - 446y_{n+1}\right)$
7	$Y^2 - 23X^2 = 641203684$	$\left(8924y_{n+1} - 18x_{n+3},\ 92x_{n+3} - 42798y_{n+1}\right)$
8	$Y^2 - 23X^2 = 3686918400$	$\left(12y_{n+2} - 518y_{n+1},\ 2484y_{n+1} - 46y_{n+2}\right)$
9	$Y^2 - 23X^2 = 5149497600$	$\left(21399y_{n+1} - 9y_{n+3},\ 46y_{n+3} - 102626y_{n+1}\right)$
10	$Y^2 - 23X^2 = 12100$	$\left(186x_{n+3} - 8924x_{n+2},\ 42798x_{n+2} - 892x_{n+3}\right)$
11	$Y^2 - 23X^2 = 484$	$\left(18y_{n+2} - 892x_{n+2},\ 4278x_{n+2} - 892y_{n+2}\right)$
12	$Y^2 - 23X^2 = 69696$	$\left(21399x_{n+2} - 93y_{n+3},\ 446y_{n+3} - 102626x_{n+2}\right)$
13	$Y^2 - 23X^2 = 69696$	$\left(4462y_{n+2} - 446x_{n+3},\ 2139x_{n+3} - 21399y_{n+2}\right)$
14	$Y^2 - 23X^2 = 484$	$\left(8924y_{n+3} - 42798x_{n+3},\ 205252x_{n+3} - 42798y_{n+3}\right)$
15	$Y^2 - 23X^2 = 6400900$	$\left(42798y_{n+2} - 892y_{n+3},\ 4278y_{n+3} - 205252y_{n+2}\right)$

Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below

[Mallika, 6(7): July 2019]
DOI- 10.5281/zenodo.3351447
ISSN 2348 – 8034
Impact Factor- 5.070

Table 3: Parabola

S.NO	Parabola	(X,Y)
1	$11Y-23X^2=484$	$\left(4y_{n+1}-18x_{n+1},\; 92x_{2n+2}-18y_{2n+2}+22\right)$
2	$55Y-23X^2=12100$	$\left(4x_{n+2}-186x_{n+1},\; 892x_{2n+2}-18x_{2n+3}+110\right)$
3	$1320Y-23X^2=6969600$	$\left(2x_{n+3}-4462x_{n+1},\; 21399x_{2n+2}-9x_{2n+4}+2640\right)$
4	$132Y-23X^2=69696$	$\left(2y_{n+2}-446x_{n+1},\; 2139x_{2n+2}-9y_{2n+3}+264\right)$
5	$12661Y-23X^2=641203684$	$\left(4y_{n+3}-42798x_{n+1},\; 205252x_{2n+2}-18y_{2n+4}+25322\right)$
6	$132Y-23X^2=69696$	$\left(93y_{n+1}-9x_{n+2},\; 46x_{2n+3}-446y_{2n+2}+264\right)$
7	$12661Y-23X^2=641203684$	$\left(8924y_{n+1}-18x_{n+3},\; 92x_{2n+4}-42798y_{2n+2}+25322\right)$
8	$1265Y-23X^2=3686918400$	$\left(12y_{n+2}-518y_{n+1},\; 2484y_{2n+2}-46y_{2n+3}+2530\right)$
9	$30360Y-23X^2=5149497600$	$\left(21399y_{n+1}-9y_{n+3},\; 46y_{2n+4}-102626y_{2n+2}+60720\right)$
10	$55Y-23X^2=12100$	$\left(186x_{n+3}-8924x_{n+2},\; 42798x_{2n+3}-892x_{2n+4}+110\right)$
11	$11Y-23X^2=484$	$\left(18y_{n+2}-892x_{n+2},\; 4278x_{2n+3}-892y_{2n+3}+22\right)$
12	$132Y-23X^2=69696$	$\left(21399x_{n+2}-93y_{n+3},\; 446y_{2n+4}-102626x_{2n+4}+264\right)$

13	$132Y - 23X^2 = 69696$	$\begin{pmatrix} 4462y_{n+2} - 446x_{n+3}, \\ 2139x_{2n+4} - 21399y_{2n+3} + 264 \end{pmatrix}$
14	$11Y - 23X^2 = 484$	$\begin{pmatrix} 8924y_{n+3} - 42798x_{n+3}, \\ 205252x_{2n+4} - 42798y_{2n+4} + 22 \end{pmatrix}$
15	$1265Y - 23X^2 = 6400900$	$\begin{pmatrix} 42798y_{n+2} - 892y_{n+3}, \\ 4278y_{2n+4} - 205252y_{2n+3} + 2530 \end{pmatrix}$

Some special cases among the solutions are given below:

$$P_y^{10}(t_{3,x+1})^2 = 207P_x^6(t_{3,y})^2 + 11(t_{3,y})^2(t_{3,x+1})^2$$
$$9P_y^6(t_{3,x})^2 = 23P_x^{10}(t_{3,y+1})^2 + 11(t_{3,x})^2(t_{3,y+1})^2$$
$$P_y^{10}(t_{3,2x-2})^2 = 23(6P_{x-1}^4)^2(t_{3,y})^2 + 11(t_{3,y})^2(t_{3,2x-2})^2$$
$$36P_{y-1}^8(t_{3,x})^2 = 23P_x^{10}(t_{3,2y-2})^2 + 11(t_{3,x})^2(t_{3,2y-2})^2$$
$$9P_y^6(t_{3,2x-2})^2 = 23(36P_{x-1}^8)(t_{3,y+1})^2 + 11(t_{3,2x-2})^2(t_{3,y+1})^2$$
$$(6P_{y-1}^4)^2(t_{3,x+1})^2 = 23(3P_x^3)^2(t_{3,2y-2})^2 + 11(t_{3,x+1})^2(t_{3,2y-2})^2$$

III. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations $y^2 = 23x^2 - 11$ As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties

REFERENCES

1. *L.E.Dickson, History of theory of number, Chelsa publishing company, vol-2, (1952) New York.*
2. *L.J.Mordel, Diophantine equations, Academic Press, (1969) New York.*
3. *S.J.Telang, Number Theory, Tata McGraw Hill Publishing company Limited,(2000) New Delhi.*
4. *D.M.Burton, Elementary Number Theory, Tata McGraw Hill Publishing company Limited,(2002) New Delhi.*
5. *M.A.Gopolan, s.vidhyalakshmi and A.Kavitha, on the integral solutions of the Binary quadratic Equation* $X^2 = 4(K^2+1)Y^2 + 4^t$ *, Bulletin of Mathematics and Statistics Research, 2(1)(2014)42-46.*
6. *S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, On the binary quadratic Diophantine equation* $x^2 - 3xy + y^2 + 33x = 0$, *International Journal of Scientific Engineering and Applied Science, 1(4) (2015) 222-225.*
7. *M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha, observations on the Hyperbola* $10y^2 - 3x^2 = 13$, *Archimedes J.Math.,3(1) (2013) 31-34.*
8. *S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, Observations on the Hyperbola* $ax^2 - (a+1)y^2 = 3a - 1$, *Discovery, 4(10) (2013) 22-24.*
9. *S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, Integral points on the Hyperbola* $x^2 - 4xy + y^2 + 15x = 0$, *Diophantus J.Maths,1(7) (2014) 338-340.*

10. M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha, on the integral solutions Of the binary quadratic equation $x^2 = 15y^2 - 11^t$, scholars journal of Engineering and technology, 2(2A) (2014) 156-158.